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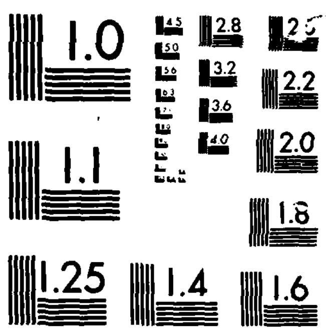

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SECONDARY BIFURCATION AND CHANGE OF TYPE FOR THREE DIMENSIONAL STANDING WAVES IN SHALLOW WATER

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ABSTRACT

This is an analysis of three dimensional surface waves which occur in a vessel of rectangular cross section and finite depth. The linearized problem has double eigenvalues for particular combinations of the parameters. It is shown that eight solution branches of finite amplitude are emitted by the double eigenvalues. Splitting of the double eigenvalues results in a secondary bifurcation. The direction of the emitted branches for the multiple and secondary bifurcation changes with the depth of the fluid. Finally it is shown that the formal solutions obtained are not uniformly valid and an additional expansion in the region of non-uniformity shows that the wave field changes type. One possibility in this region is a field of three-dimensional cnoidal standing waves.

AMS (MOS) Subject Classifications: 76B15, 35B10, 35B22, 35R35

Key Words: standing cnoidal waves, multiple eigenvalues, secondary bifurcation

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SECONDARY BIFURCATION AND CHANGE OF TYPE FOR THREE DIMENSIONAL STANDING WAVES IN SHALLOW WATER

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1. Introduction

Three dimensional irrotational standing waves of finite amplitude in a basin of rectangular cross section and finite depth are considered in this paper. The governing equation and boundary conditions in dimensionless form for the potential and the wave height are

$$\frac{\partial \phi}{\partial x^2} + \frac{\partial \phi}{\partial y^2} + \xi^2 \frac{\partial \phi}{\partial z^2} = 0 \tag{1.1}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \textit{on solid boundary} \tag{1.2}$$

and on $y = \epsilon\eta(x,z,t)$

$$\omega\frac{\partial \eta}{\partial t} + \epsilon\frac{\partial \eta}{\partial x}\frac{\partial \phi}{\partial x} + \epsilon\xi^2\frac{\partial \eta}{\partial z}\frac{\partial \phi}{\partial z} - \frac{\partial \phi}{\partial y} = 0 \tag{1.3}$$

$$\omega\frac{\partial \phi}{\partial t} + \frac{1}{2}\epsilon\Big[(\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2 + \xi^2(\frac{\partial \phi}{\partial z})^2\Big] + \eta = 0 \tag{1.4}$$

where $\xi = \frac{2a}{2b}$, $2a$ is the vessel length in the x direction, $2b$ is the vessel length in the z direction, $\delta = \frac{h}{2a}$, where h is the vessel still water depth, $\epsilon = \frac{H}{h}$ where H is a measure of the wave height.

The linearized problem for (1.1)-(1.4) has eigenvalues (bifurcation points, linear natural frequencies)

$$\sigma_0 = \sqrt{\lambda_{mn}\tanh\lambda_{mn}\delta} \tag{1.5}$$

where

$$\lambda_{mn} = \pi\sqrt{m^2 + \xi^2 n^2} \tag{1.6}$$

The solutions for finite but small amplitude emitted by the *simple* linear eigenvalues have been found by Verma & Keller (1962) using a formal perturbation expansion. However at particular combinations of (m,n) and ξ there are double eigenvalues. It is shown in section 2 that eight solution branches are emitted by these double eigenvalues. This is done using a formal perturbation expansion similar to Verma & Keller.

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In an interesting discovery Bauer, Keller, & Reiss (1975) observed that as a multiple bifurcation point is "split" (by varying an auxiliary parameter (in this case ξ)) into primary bifurcation points a secondary bifurcation may occur.

In the standing wave problem there are four $+\epsilon$ branches emitted by the double eigenvalues. Splitting a double eigenvalue results in two simple bifurcation points with a single $+\epsilon$ branch emitted from each. Rather than vanish, the other two branches from the double eigenvalue slowly depart by creeping up a primary branch. Bauer, Keller, & Reiss developed a perturbation method to analyze this phenomena. Subsequently this theory has been successfully applied by Matkowsky, Putnick & Reiss (1978) to the buckling of rectangular plates, Kriegsmann & Reiss (1978) to the theory of magnetohydrodynamic equilibria, and has been extended to the bifurcation from triple eigenvalues, which results in secondary and tertiary bifurcation by Reiss (1983).

This theory is applied in Section 3 to show that secondary bifurcation occurs in the neighborhood of the double eigenvalues. The secondary bifurcation points for a perturbed square cross section are found as a function of δ and the mode numbers and solutions along the secondary branches are derived. It is found that the jump to a secondary branch produces interesting irregular wave forms.

The solution of Verma & Keller, the finite amplitude solutions found in section 2 emitted by the double eigenvalues, and the secondary bifurcation phenomena elucidated in section 3 are valid for a small range of amplitude only. It is shown that when $\epsilon = O(\delta^2)$ the higher order terms are no longer of higher order and the expansion breaks down.

In section 4 a separate analysis is performed with the assumption that $\epsilon = 0(\delta^2)$. As a first approximation weakly three dimensional waves are considered. This is done by looking in the region of the $\xi - \delta - \epsilon$ parameter space where the triple balance holds

$$\xi^2 = 0(\epsilon)$$

$$\delta^2 = 0(\epsilon)$$

This analysis results in a field of standing K-P waves. A set of two non-interacting (to first order) solutions of the K-P equation (Kadomtsev-Petviashvili 1970). The K-P equation, which is rich in solutions, has been studied in some detail by Dubrovin (1981) and Segur & Finkel

(1984). Therefore as the amplitude increases beyond $\epsilon = 0(\delta^2)$ the wave field changes type. The fact that there are multiple solutions when $\epsilon << \delta^2$ and the fact that the K-P equation is rich in solution possibilities suggests that the region for $\epsilon \sim 0(\delta^2)$ will result in a "rats nest" of branches of solutions.

The results with the K-P equations are for weakly three dimensional waves. A further analysis is performed in section 4 with this assumption relaxed. This analysis results in a wave equation to leading order. The solvability condition at the next order results in a functional differential equation for the leading order term. A complete solution is not found but it is shown that one solution is a set of four oblique travelling cnoidal waves which combine to form a three dimensional standing wave. Distributions of the wave height are shown for these waves. It is expected that this equation will yield other interesting possibilities.

2. Primary Bifurcation When $\epsilon < 0(\delta^2)$

Linearizing the set of equations about the still water level results in a linear problem which has eigenvalues (linear natural frequencies)

$$\sigma_0 = \sqrt{\lambda_{mn} \tanh \lambda_{mn}\delta} \tag{2.1}$$

and eigenfunctions

$$\eta_1 = \cos \alpha_m \bar{x} \cos \beta_n \bar{z} \sin t \tag{2.2}$$

$$\phi_1 = \sigma_0^{-1} \frac{\cosh \lambda_{mn}(y+\delta)}{\cosh \lambda_{mn}\delta} \cos \alpha_m \bar{x} \cos \beta_n \bar{z} \cos t \tag{2.3}$$

where $\lambda_{mn} = \sqrt{\alpha_m^2 + \xi^2\beta_n^2}$, $\alpha_m = m\pi$, $\beta_n = n\pi$, $\bar{x} = x + \frac{1}{2}$, $\bar{z} = z + \frac{1}{2}$, and m, n are the mode numbers in the x,z directions.

The solutions which bifurcate from the linear eigenvalues (2.1) were first found by Verma & Keller (1962) using a perturbation expansion in the amplitude. They found that the natural frequency for the three dimensional standing wave has the following form as $\epsilon \to 0$

$$\omega = \sigma_0(1 + \epsilon^2 \frac{\sigma_2}{\sigma_0} + \cdots) \tag{2.4}$$

where

$$\frac{\sigma_2}{\sigma_0} = \frac{1}{256}\{9\frac{\lambda_{mn}^6}{\sigma_0^6} - 24\frac{(\alpha_m^4 + \xi^4\beta_n^4)}{\sigma_0^4} + 5\lambda_{mn}^2 - 46\sigma_0^4\}$$

$$-\frac{(3\sigma_0^4+\lambda_{mn}^2-4\alpha_m^2)^2}{64\sigma_0^2(\alpha_m \tanh 2\alpha_m\delta - 2\sigma_0^2)} - \frac{(3\sigma_0^4+\lambda_{mn}^2-4\xi^2\beta_n^2)^2}{64\sigma_0^2(\beta_n\xi \tanh 2\beta_n\xi\delta - 2\sigma_0^2)} \tag{2.5}$$

Taking $\alpha_m = \beta_n = 1$ and $\xi = \frac{1}{L}$ this agrees with the expression in the paper by Verma & Keller. The bifurcation for the frequency is sub- or supercritical depending on the value of δ. As $\delta \to \infty$ (deep water) the bifurcation is subcritical and as $\delta \to 0$ (shallow water) it is supercritical. Figure 2 of the paper by Verma & Keller shows, for the first mode, that as the vessel cross-section departs from being a square the critical depth δ^* (the point where the bifurcation changes from sub- to supercritical) increases. The same phenomena occurs for the higher modes as well with the critical depth being smaller as the mode number increases (for fixed ξ). Consequently when a lower mode is supercritical it may be that a higher mode, with all other parameters being equal, may bifurcate subcritically suggesting the possibility of an intersection of the branches at finite amplitude.

It can be shown however that the asymptotic solution obtained by Verma & Keller is not uniformily valid. Taking the limit as $\delta \to 0$ it is found that

$$\frac{\sigma_2}{\sigma_0} \to \frac{k}{\lambda_{mn}^2\delta^4}$$

where k is a constant of $0(1)$. Therefore the solution is valid for $\epsilon < 0(\delta^2)$ only. In section 4 the equations will be reanalyzed for the region $\epsilon = 0(\delta^2)$ and it will be shown that the solutions change type in this region.

Inspection of (2.1) also shows that the linearized problem has double eigenvalues at particular combinations of α_m, β_n, and ξ. For example, when $\xi = 1$ every pair (α_m, β_n) such that $m \neq n$ is a double eigenvalue and, for example, when $\xi = \frac{1}{2}$ $(m,n) = (1,4)$ and $(2,2)$ share the same eigenvalue. In fact every rational ξ will have an infinite set of double eigenvalues. It will now be shown that these double eigenvalues emit multiple branches of solutions. For brevity the problem of a square vessel ($\xi = 1$) will be considered and it is expected that the analysis at other double eigenvalues will result in a similar conslusion. When $\xi = 1$ the bifurcation points occur at (2.1) with

$$\lambda_{mn} = \sqrt{\alpha_m^2 + \beta_n^2} \tag{2.6}$$

Therefore any pair (m,n) such that $m \neq n$ will result in a double eigenvalue. The analysis proceeds by formally expanding the frequency, potential, and wave height in a regular perturbation series. The leading term in the frequency expansion is given by (2.1) with (2.6) and the

leading term for each of the dependent variables is

$$\eta_1 = [A_{11} \cos \alpha_m \bar{x} \cos \beta_n \bar{z} + A_{12} \cos \beta_n \bar{x} \cos \alpha_m \bar{z}] \sin t \tag{2.7a}$$

$$\phi_1 = \frac{1}{\sigma_0}[A_{11} \cos \alpha_m \bar{x} \cos \beta_n \bar{z} + A_{12} \cos \beta_n \bar{x} \cos \alpha_m \bar{z}] \frac{\cosh \lambda_{mn}(y+\delta)}{\cosh \lambda_{mn}\delta} \cos t \tag{2.7b}$$

A normalization for the coefficients is chosen such that

$$A_{11}^2 + A_{12}^2 = 1 \tag{2.8}$$

The relative magnitudes of A_{11} and A_{12} are determined at higher order.

Proceeding in the usual way results in the following set of bifurcation equations after application of the double solvability condition at the third order.

$$[a_1 A_{11}^2 + a_2 A_{12}^2 + 2\sigma_2] A_{11} = 0 \tag{2.9a}$$

$$[a_2 A_{11}^2 + a_1 A_{12}^2 + 2\sigma_2] A_{12} = 0 \tag{2.9b}$$

which along with (2.8) form a set of three equations for the three unknowns: σ_2, A_{11}, A_{12}. The coefficients a_1 and a_2 are given by

$$a_1 = \frac{\sigma_0}{128}\left[-9\frac{\lambda_{mn}^6}{\sigma_0^8} + 24\frac{(\alpha_m^4 + \beta_n^4)}{\sigma_0^4} - 5\lambda_{mn}^2 + 46\sigma_0^4\right]$$
$$+ \frac{(3\sigma_0^4 - 3\alpha_m^2 + \beta_n^2)^2}{32\sigma_0[\alpha_m \tanh 2\alpha_m\delta - 2\sigma_0^2]} + \frac{(3\sigma_0^4 - 3\beta_n^2 + \alpha_m^2)^2}{32\sigma_0[\beta_n \tanh 2\beta_n\delta - 2\sigma_0^2]} \tag{2.10}$$

$$a_2 = \frac{(3\sigma_0^4 - \lambda_{mn}^2 - 4\alpha_m\beta_n)^2}{16\sigma_0[\sqrt{2}(\alpha_m + \beta_n)\tanh[\sqrt{2}(\alpha_m + \beta_n)\delta] - 4\sigma_0^2]}$$
$$+ \frac{(3\sigma_0^4 - \lambda_{mn}^2 + 4\alpha_m\beta_n)^2}{16\sigma_0[\sqrt{2}(\alpha_m - \beta_n)\tanh[\sqrt{2}(\alpha_m - \beta_n)\delta] - 4\sigma_0^2]}$$
$$+ \frac{(3\sigma_0^4 - \lambda_{mn}^2)^2}{8\sigma_0[\sqrt{2}\lambda_{mn}\tanh[\sqrt{2}\lambda_{mn}\delta] - 4\sigma_0^2]}$$
$$+ \frac{\sigma_0}{16}\{11\sigma_0^4 + 3\frac{\lambda_{mn}^4}{\sigma_0^4} - 2\lambda_{mn}^2 - 3\frac{(\alpha_m^4 + \beta_n^4)}{\sigma_0^4}\} \tag{2.11}$$

The three equations (2.8)-(2.9) have the following set of eight solutions

$$Pure\ \#1: \quad A_{11} = \pm 1 \quad A_{12} = 0 \quad \sigma_2^p = -\frac{1}{2}a_1 \tag{2.13a}$$

$$Pure\ \#2: \quad A_{11} = 0 \quad A_{12} = \pm 1 \quad \sigma_2^p = -\frac{1}{2}a_1 \tag{2.13b}$$

$$Mixed: \quad A_{11} = \pm\frac{1}{\sqrt{2}} \quad A_{12} = \pm\frac{1}{\sqrt{2}} \quad \sigma_2^m = -\frac{1}{4}(a_1 + a_2) \tag{2.13c}$$

The pure modes correspond to the modes of the simple eigenvalues that coalesce to form the double point. They share the same natural frequency and are spatially (horizontally) symmetric. When the relevant parameters are substituted the amplitude correction to the natural frequency $\sigma_2^p = -\frac{1}{2}a_1$ agrees with the correction found for the simple eigenvalues (equation (2.5) with $\xi = 1$). The other four solutions are mixed modes. The leading terms are proportional to the sum or difference of the two pure eigenfunctions. The amplitude correction of the frequency for the mixed mode solutions σ_2^m differs from that for the pure modes by an amount $\frac{1}{4}(a_1 - a_2)$. The bifurcation for either the pure or mixed modes will be sub- or super-critical depending on the value of δ.

Figures 1a,b,c,d, and e are bifurcation diagrams for the natural frequency for the multiple eigenvalues occuring in a square cross-section with mode numbers $m = 1$ and $n = 2$. The five frames show the effect of the parameter δ on the behavior of the solutions. Figure 1a corresponds to infinite depth and agrees with the result in Bridges (1986). The mixed branch apparently has a higher natural frequency for the range of amplitudes considered and for all values of the depth. The remainder of the Figure 1 set correspond to decreasing values of δ. As δ decreases the branches all shift to the right and eventually (at the critical depth δ^*) shift from sub- to supercritical. At $\delta = .115$ there is the interesting property that the pure branch bifurcates subcritically and the mixed branch bifurcates supercritically.

In Bridges (1986) distributions of the wave height for the pure branches and the mixed mode branches for $\delta \to \infty$ are shown. It is expected that, qualitatively, the distributions for finite δ (and suitably restricted ϵ) will be similar.

3. Secondary Bifurcation when $\epsilon < 0(\delta^2)$

In this section solutions which bifurcate at finite amplitude from the branches emitted by the simple eigenvalues are considered. Therefore a perturbation is added to the known primary branch solutions (Φ, Υ, ω) found by Verma & Keller,

$$\phi = \epsilon'\Phi + \phi' \tag{3.1a}$$

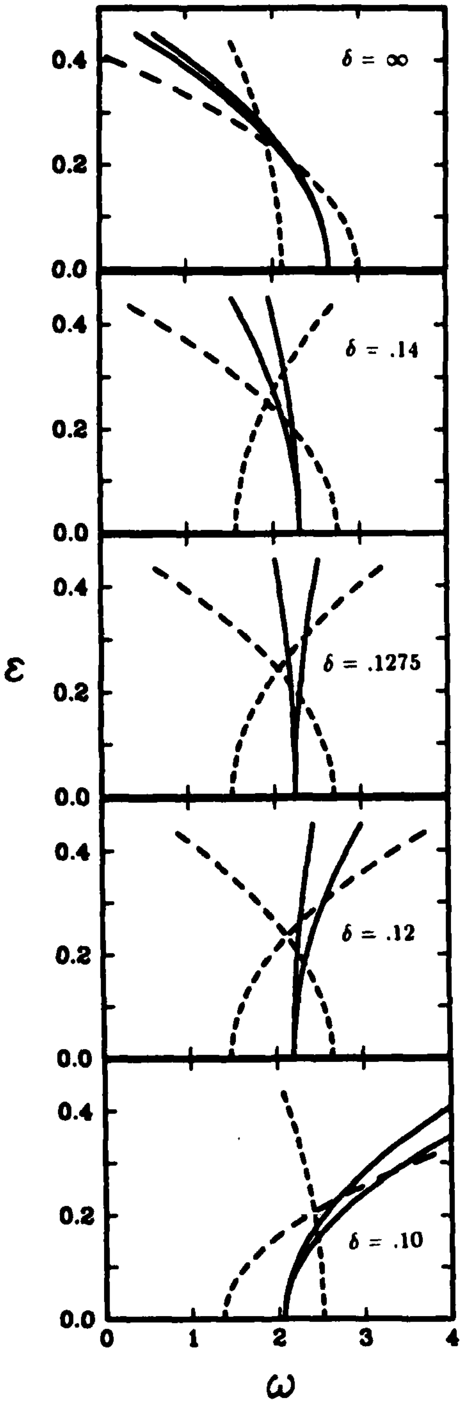

Figure 1. Effect of δ on the bifurcation from the double eigenvalues for $\xi = 1$ and (m,n) ranging over 1,2. The dashed lines correspond to the simple modes $\sigma_{1,1}$ (left branch) and $\sigma_{2,2}$ (right branch) and the solid lines are for pure (left branch) and mixed (right branch) modes emitted by the double point $\sigma_{1,2}, \sigma_{2,1}$.

$$\eta = \epsilon' \Upsilon + \eta' \tag{3.1b}$$

These expressions are substituted into the governing equations and boundary conditions which are then linearized about the known primary branch solutions. The linear problem to be solved for the points, if they exist, of secondary bifurcation is

$$\frac{\partial^2 \phi'}{\partial x^2} + \frac{\partial^2 \phi'}{\partial y^2} + \xi^2 \frac{\partial^2 \phi'}{\partial z^2} = 0 \tag{3.2}$$

and on $y = \epsilon' \Upsilon(x, z, t)$,

$$\omega \frac{\partial \eta'}{\partial t} + \epsilon' \frac{\partial \Phi}{\partial x}\frac{\partial \eta'}{\partial x} + \epsilon' \frac{\partial \Upsilon}{\partial x}\frac{\partial \phi'}{\partial x} + \epsilon' \xi^2 \left[\frac{\partial \Phi}{\partial z}\frac{\partial \eta'}{\partial z} + \frac{\partial \Upsilon}{\partial z}\frac{\partial \phi'}{\partial z}\right] - \frac{\partial \phi'}{\partial y} = 0 \tag{3.3}$$

and

$$\omega \frac{\partial \phi'}{\partial t} + \epsilon' \left[\frac{\partial \Phi}{\partial x}\frac{\partial \phi'}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial \phi'}{\partial y} + \frac{\partial \Phi}{\partial z}\frac{\partial \phi'}{\partial z}\right] + \eta' = 0 \tag{3.4}$$

and in addition the normal derivative is required to vanish on the solid boundary. This is a linear differential eigenvalue problem with known nonconstant coefficients. However, the specific value of ϵ' where the secondary bifurcation takes place is sought. Therefore ϵ' is the eigenvalue. Since ϵ' appears nonlinearly it is a nonlinear in the parameter eigenvalue problem and there is the further complication that ϵ' is responsible for the size of the domain. The qualitative shape of the domain is known since $\Upsilon(x, z, t)$ is a known function, but the precise amount of $\Upsilon(x, z, t)$ is the unknown eigenvalue. This is to be constrasted with the original eigenvalue problem in Section 2 where ω was the eigenvalue, ϵ' was a variable parameter, and the *shape* of the free surface (and hence the domain) was an unknown function.

To solve this eigenvalue problem, the conjecture of Bauer, Keller, & Reiss (1975), that a secondary bifurcation may occur in the neighborhood of a multiple eigenvalue, is used. For brevity, the analysis is undertaken in the neighborhood of $\xi = 1$. It is expected that a similar analysis will hold in the neighborhood of other values of ξ at which double eigenvalues occur.

It was shown in Section 2 that for $\xi = 1$ there is a double eigenvalue for every pair (m, n) such that $m \neq n$. At the double eigenvalue the bifurcation points are given by (2.1) with (2.6). In the neighborhood of $\xi = 1$ this double eigenvalue splits into two primary branches emitted by the bifurcation points

$$\sigma_{m,n} = \sqrt{\pi \sqrt{m^2 + \xi^2 n^2} \tanh \pi \delta \sqrt{m^2 + \xi^2 n^2}} \tag{3.5}$$

$$\sigma_{n,m} = \sqrt{\pi\sqrt{n^2 + \xi^2 m^2}\tanh[\pi\delta\sqrt{n^2 + \xi^2 m^2}]} \tag{3.6}$$

A measure of the neighborhood of $\xi = 1$ is given by the small parameter μ defined by

$$\xi = 1 + \frac{1}{2}\tau\mu^2 \quad where \quad \tau = sign(\xi - 1) = \pm 1 \tag{3.7}$$

Following the conjecture of Bauer, Keller, & Reiss that the secondary bifurcation disapears at the double eigenvalue, the point ϵ' on the primary branches where the secondary bifurcation will take place is expressed as

$$\epsilon'(\mu) = b_0\mu + b_1\mu^2 + \cdots \tag{3.8}$$

The solutions previously obtained by Verma & Keller for the primary branches emitted by simple eigenvalues are recast using (3.7)-(3.9) as expansions in the small parameter μ,

$$\Phi = \psi_0 + \mu\psi_1 + \cdots \tag{3.9a}$$

$$\Upsilon = \Upsilon_0 + \mu\Upsilon_1 + \cdots \tag{3.9b}$$

$$\omega = \omega_0 + \mu^2\omega_2 + \cdots \tag{3.10}$$

A separate analysis is undertaken for each of the primary branches σ_{mn} and σ_{nm}. The necessary details for the analysis along the σ_{mn} branch will be given and the result only will be stated for the σ_{nm} branch.

Substituting the expressions (3.7)-(3.10) into the linear eigenvalue problem and postulating that

$$\phi' = \mu\phi_1' + \mu^2\phi_2' + \cdots \tag{3.11a}$$

$$\eta' = \mu\eta_1' + \mu^2\eta_2' + \cdots \tag{3.11b}$$

Expanding the free surface boundary conditions in a Taylor series, and equating terms proportional to like powers of μ to zero results in a sequence of boundary value problems. The fact that ω_0 is a double eigenvalue results in the leading term in the set

$$\omega_0 = \lambda_0 \tanh \lambda_0\delta \tag{3.12}$$

where

$$\lambda_0 = \pi\sqrt{m^2 + n^2} \tag{3.13}$$

$$\eta_1'(x,z,t) = [A_{11}\cos\alpha_m\bar{x}\cos\beta_n\bar{z} + A_{12}\cos\beta_n\bar{x}\cos\alpha_m\bar{z}]\sin t \tag{3.14}$$

$$\phi_1'(x,y,z,t) = \frac{1}{\omega_0}\frac{\cosh\lambda_{mn}(y+\delta)}{\cosh\lambda_{mn}\delta}[A_{11}\cos\alpha_m\bar{x}\cos\beta_n\bar{z} + A_{12}\cos\beta_n\bar{x}\cos\alpha_m\bar{z}]\cos t \tag{3.15}$$

and the normalization is taken to be

$$A_{11}^2 + A_{12}^2 = 1 \tag{3.16}$$

The problem is carried in the usual way to higher order. At third order application of the double solvability condition results in the equations

$$b_0^2 A_{11} = 0 \tag{3.17}$$

$$\left[\frac{\tau}{\omega_0^2}(\alpha_m^2 - \beta_n^2)\{\frac{\omega_0^2}{2\lambda_0^2} + (\lambda_0^2 - \omega_0^4)\frac{\delta}{2\lambda_0^2}\} + a_3 b_0^2\right]A_{12} = 0 \tag{3.18}$$

which with (3.16) form a set of three equations for the three unknowns b_0, A_{11}, and A_{12}. The term a_3 is given by

$$\begin{aligned} a_3(\delta) = {} & \frac{1}{128}\{27\lambda_0^2 + 14\omega_0^4 - 24\frac{\lambda_0^4}{\omega_0^4} - 9\frac{\lambda_0^6}{\omega_0^8}\} + \frac{3(\alpha_m^4 + \beta_n^4)}{8\omega_0^4} \\ & + \frac{(3\omega_0^4 + \lambda_0^2 - 4\alpha_m^2)^2}{32\omega_0^2(\alpha_m\tanh 2\alpha_m\delta - 2\omega_0^2)} + \frac{(3\omega_0^4 + \lambda_0^2 - 4\beta_n^2)^2}{32\omega_0^2(\beta_n\tanh 2\beta_n\delta - 2\omega_0^2)} \\ & - \frac{[\omega_0^4 - (\alpha_m - \beta_n)^2][4\omega_0^4 - \lambda_0^2 + 8\alpha_m\beta_n]}{16\omega_0^2\{-4\omega_0^2 + \sqrt{2}(\alpha_m - \beta_n)\tanh[\sqrt{2}(\alpha_m - \beta_n)\delta]\}} \\ & - \frac{[\omega_0^4 - (\alpha_m + \beta_n)^2][4\omega_0^4 - \lambda_0^2 - 8\alpha_m\beta_n]}{16\omega_0^2\{-4\omega_0^2 + \sqrt{2}(\alpha_m + \beta_n)\tanh[\sqrt{2}(\alpha_m + \beta_n)\delta]\}} \\ & - \frac{[\omega_0^4 - \lambda_0^2][4\omega_0^4 - \lambda_0^2]}{8\omega_0^2\{-4\omega_0^2 + \sqrt{2}\lambda_0\tanh[\sqrt{2}\lambda_0\delta]\}} \end{aligned} \tag{3.19}$$

For sufficiently large δ it has been shown by numerical evaluation for m, n ranging over 1 to 10 that the expression for a_3 is positive definite. In the limit as $\delta \to 0$ however $a_3 \to -\frac{9}{128}\lambda_0^2\delta^{-4}$. Therefore as $\delta \to 0$ a_3 will eventually become negative. For example when $(m,n) = (1,2)$ a_3 changes sign when $\delta \sim .075$. When a_3 changes sign this means that the secondary bifurcation will "jump" from one branch to another which would be a discontinuous phenomena. It is more likely that this behavior is a ramification of the non-uniformity in δ of the solution.

The solutions to (3.16)-(3.18) are

$$\textit{Case I}: \quad b_0 = 0,\ A_{11} = 1,\ A_{12} = 0 \tag{3.20a}$$

$$\textit{Case II}: \quad b_0 = \pm\sqrt{-\frac{(\alpha_m^2 - \beta_n^2)\tau}{2a_3\omega_0^4}\{\frac{\omega_0^2}{2\lambda_0^2} + (\lambda_0^2 - \omega_0^4)\frac{\delta}{2\lambda_0^2}\}},\ A_{11} = 0,\ A_{12} = 1 \tag{3.20b}$$

The solution in (3.20a) shows that the basic solution bifurcates from the primary branch. In (3.20b) the solution for the secondary bifurcation on branch σ_{mn} is given. The $\pm$ sign shows that the bifurcation takes place in both the upper and lower ϵ half-planes. The jump to the $A_{12} \neq 0$ solution is often referred to as mode jumping because the solution acquired on the secondary branch is qualitatively different from that on the primary branch. The radical in (3.20b), when $a_3 > 0$, requires that $(\alpha_m - \beta_n)\tau < 0$ for secondary bifurcation to occur on branch σ_{mn}.

A similar analysis for the σ_{nm} branch results in the bifurcation equations

$$\left[-\frac{\tau}{\omega_0^2}(\alpha_m^2 - \beta_n^2)\left\{\frac{\omega_0^2}{2\lambda_0^2} + (\lambda_0^2 - \omega_0^4)\frac{\delta}{2\lambda_0^2}\right\} + a_3 b_0^2\right]A_{11} = 0 \tag{3.21a}$$

$$b_0^2 A_{12} = 0 \tag{3.21b}$$

which gives the points of secondary bifurcation on that branch. For convenience define $\epsilon_{m,n}$ to be the point of secondary bifurcation on the σ_{mn} branch and $\epsilon_{n,m}$ to be the that on the σ_{nm} branch, then

$$\epsilon_{m,n}(\mu) = b_{m,n}\mu + 0(\mu^2) \tag{3.22a}$$

$$\epsilon_{n,m}(\mu) = b_{n,m}\mu + 0(\mu^2) \tag{3.22b}$$

Retaining the positive branch only for brevity the bifurcation equations on each branch show that

$$b_{m,n} = \sqrt{-\frac{\tau(\alpha_m^2 - \beta_n^2)}{2\lambda_0^2 a_3(\delta)}[1 + \frac{(\lambda_0^2 - \omega_0^4)}{\omega_0^2}\delta]} \tag{3.23a}$$

$$b_{m,n} = \sqrt{\frac{\tau(\alpha_m^2 - \beta_n^2)}{2\lambda_0^2 a_3(\delta)}[1 + \frac{(\lambda_0^2 - \omega_0^4)}{\omega_0^2}\delta]} \tag{3.23b}$$

A secondary bifurcation takes place on one, and only one at a time, branch. Noting that $\tau = sign(\xi - 1)$ the branch on which the secondary bifurcation takes place is

$\xi - 1$	$(\alpha_m - \beta_n)$	*Branch*
+	+	(n,m)
+	−	(m,n)
−	+	(m,n)
−	−	(n,m)

In summary, as ξ departs from $\xi = 1$, the split primary bifurcation points given by (3.5), (3.6) move away from the double point. When $\xi > 1$ they both move to the right and when $\xi < 1$ they both move to the left. However in all four cases given in the table the secondary bifurcation takes place on the branch which is emitted, after splitting, by the *largest*, in magnitude, of the two bifurcation points, regardless of the sign of τ.

It has been shown that in the neighborhood of a square cross section ($\xi = 1$) a secondary bifurcation will occur on one of the two branches emitted from the simple eigenvalues which result from the splitting of the double eigenvalues. By expanding in the neighborhood of this point an asymptotic representation of the solution along the secondary branch may be found. A small parameter ν is defined as a measure of the distance from the point of secondary bifurcation. It is assumed that the parameters are such that the secondary bifurcation point occurs on the σ_{mn} branch. A similar analysis may be performed for a bifurcation from the σ_{nm} branch.

A perturbation is added to the known primary branch solution

$$\phi = \epsilon\Phi + \phi' \tag{3.24a}$$

$$\eta = \epsilon\Upsilon + \eta' \tag{3.24b}$$

$$\omega = \sigma + \Omega \tag{3.24c}$$

and the unknown solutions on the secondary branch are expressed as a regular perturbation series in ν and μ

$$\phi' = \nu(\mu\phi_{11} + \mu^2\phi_{12} + \cdots) + \nu^2(\mu\phi_{21} + \mu^2\phi_{22} + \cdots) + \cdots \tag{3.25}$$

$$\eta' = \nu(\mu\eta_{11} + \mu^2\eta_{12} + \cdots) + \nu^2(\mu\eta_{21} + \mu^2\eta_{22} + \cdots) + \cdots \tag{3.26}$$

$$\Omega = \nu(\Omega_{10} + \mu\Omega_{11} + \mu^2\Omega_{12} + \cdots) + \nu^2(\Omega_{20} + \mu\Omega_{21} + \mu^2\Omega_{22} + \cdots) + \cdots \tag{3.27}$$

The substitution of expressions (3.24)-(3.27) into the governing equations and boundary conditions results in a set of boundary value problems for the unknowns ϕ_{ij} and η_{ij}. The analysis although straightforward is lengthy and the details will be omitted.

The first order in ν problem results in $\Omega_{1j} = 0$ for all j and

$$\phi_{11} = \frac{1}{\omega_0}\frac{\cosh\lambda_0(y+\delta)}{\cosh\lambda_0\delta}\cos\beta_n\bar{x}\cos\alpha_m\bar{z}\cos t \tag{3.28a}$$

$$\eta_{11} = \cos\beta_n \bar{x} \cos\alpha_m \bar{z} \sin t \tag{3.28b}$$

and the higher order terms (in ν) are omitted for brevity. The problem of order ν^2 results in $\Omega_{20} = \Omega_{21} = 0$, $\phi_{21} = \eta_{21} = 0$, and

$$\begin{aligned} \frac{\Omega_{22}}{\omega_0} = \frac{1}{32}\left(2\lambda_0^2 - 11\omega_0^4 - 6\frac{\alpha_m^2\beta_n^2}{\omega_0^4}\right) + \frac{[3\omega_0^4 - \lambda_0^2]^2}{16\omega_0^2[4\omega_0^2 - \sqrt{2}\lambda_0 \tanh(\sqrt{2}\lambda_0\delta)]} \\ + \frac{[3\omega_0^4 - \lambda_0^2 + 4\alpha_m\beta_n]^2}{32\omega_0^2[4\omega_0^2 - \sqrt{2}(\alpha_m - \beta_n)\tanh(\sqrt{2}(\alpha_m - \beta_n)\delta)]} \\ + \frac{[3\omega_0^4 - \lambda_0^2 - 4\alpha_m\beta_n]^2}{32\omega_0^2[4\omega_0^2 - \sqrt{2}(\alpha_m + \beta_n)\tanh(\sqrt{2}(\alpha_m + \beta_n)\delta)]} \end{aligned} \tag{3.29}$$

and the other higher order terms are omitted. The result (3.29) provides an expression for the frequency along the secondary branches. The complete expression for the natural frequency in the neighborhood of the double eigenvalue is

$$\omega = \omega_0 + \mu^2\omega_2 + \mu^2\nu^2\Omega_{22} + 0(\mu^3, \nu^3) \tag{3.30}$$

Since Ω_{22} is proportional to quadratic terms the sign of Ω_{22} determines whether the bifurcation is sub- or supercritical. Although no proof has been undertaken the following points regarding the $sign[\Omega_{22}]$ are made based on numerical evaluation of (3.29). As $\delta \to \infty$ the $sign[\Omega_{22}] < 0$ for all mode numbers. As δ is decreased a critical value of δ is reached where Ω_{22} changes sign and this critical value differs for different mode numbers. For example when $(m,n) = (1,2)$ Ω_{22} changes sign from $+$ to $-$ when $\delta \sim 0.1155$ resulting in a shift of the secondary bifurcation from sub- to supercritical. Since the critical value of δ on the primary branch is slightly different from the critical value for the secondary branch there is a small range of δ where the primary branch is supercritical and the secondary branch is subcritical.

Figures 2a,b,c,d, and e give an illustration of the secondary bifurcation phenomena for various δ when $(m,n) = (1,2)$. Figure 1a for $\delta = .20$ is similar to the infinite depth result obtained in Bridges (1986). The remainder of the sequence in Figure 2 shows the shifting of the branches to the right as δ is decreased. In Figure 2d the secondary bifurcation is almost vertical as $\Omega_{22} \sim 0$ here, and in Figure 2e the secondary bifurcation has shifted to supercritical. An example of the distribution of the wave height (for $\delta \to \infty$) as the solution shifts from the primary to the secondary pranch is shown in Bridges (1986). The wave field becomes more complex as the solution on the secondary branch is acquired. The addition of the finite depth is not expected to significantly alter qualitatively this distribution for suitably restricted amplitude.

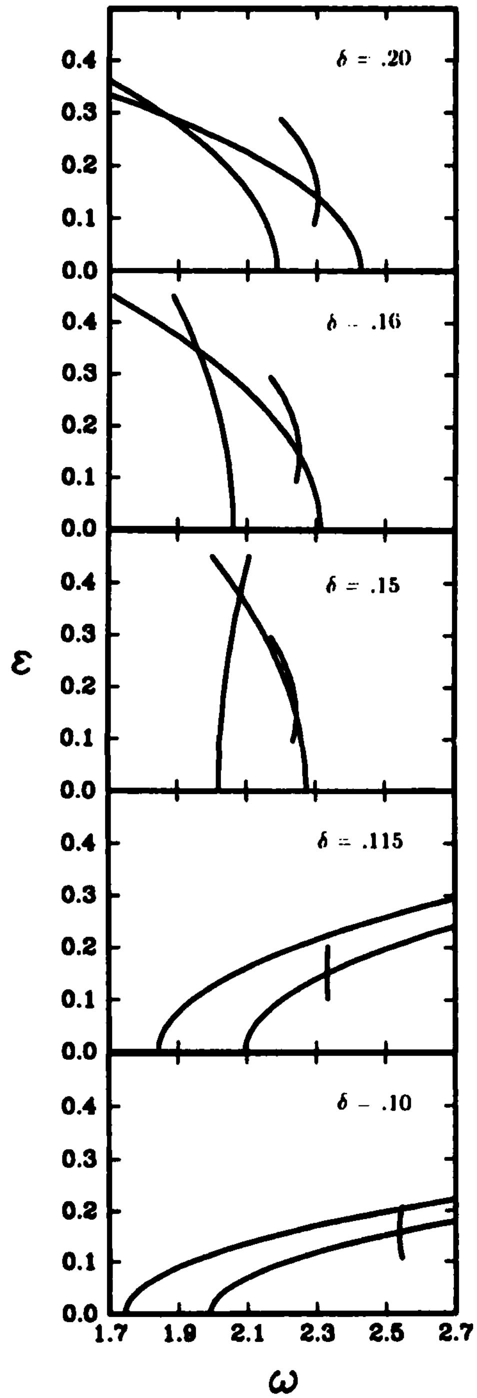

Figure 2. Effect of δ on the secondary bifurcation when $(m,n) = (2,1)$ and $\xi = \frac{7}{9}$. After the splitting the secondary bifurcation occurs on the $\sigma_{2,1}$ branch. As δ is decreased the primary and then the secondary bifurcation shifts from sub- to supercritical.

4. Change of Type When $\epsilon = 0(\delta^2)$

The *solutions obtained in* Sections 2 & 3 are not uniformily valid in the ϵ - δ plane. The higher order terms are no longer of higher order when $\epsilon = 0(\delta^2)$. Therefore in this section a separate analysis is performed for that region of the ϵ - δ plane where $\epsilon = 0(\delta^2)$. This is done by taking $\delta = \sqrt{\tau\epsilon}$ where $\tau = 0(1)$ and carries the sign of ϵ (and is different from the τ used in Section 3). The governing equations are (1.1)-(1.4) but with the scaling modified so that δ appears explicitly in the equation (y is scaled with h instead of $2a$). With this scaling the linear natural frequency has a finite non-zero limit as $\delta \to 0$.

Before proceeding to the fully three dimensional problem it is useful to analyze *weakly* three dimensional waves by (following Ablowitz & Segur (1979)) considering the region of parameter space where $\xi^2 = 0(\epsilon)$, or

$$\xi = \sqrt{\gamma\epsilon} \quad \textit{and} \quad \delta = \sqrt{\tau\epsilon} \tag{4.1}$$

where $\gamma = 0(1)$ and carries the sign of ϵ. When the relations (4.1) are substituted into the governing equations and boundary conditions and a regular expansion in ϵ is sought the leading order problem is a wave equation

$$\sqcup\psi_0(x,z,t) = 0 \tag{4.2}$$

where $\sqcup$ is the D'Alembertian

$$\sqcup \equiv \omega_0^2\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \tag{4.3}$$

With the additional requirement that $\frac{\partial\psi_0}{\partial n}$ vanish on the vertical boundaries (4.2) has the general solution

$$\omega_0 = \alpha_m \tag{4.4}$$

$$\psi_0(x,z,t) = f(\varsigma, z) + f(\chi, z) \tag{4.5}$$

where $\alpha_m = m\pi$, $\varsigma = t + \alpha_m\bar{x}$, $\chi = t - \alpha_m\bar{x}$, and $\bar{x} = x + \frac{1}{2}$. The leading order wave height is

$$\eta_0(x,z,t) = -\omega_0[\frac{\partial}{\partial\varsigma}f(\varsigma,z) + \frac{\partial}{\partial\chi}f(\chi,z)] \tag{4.6}$$

the unknown function f is found through application of the solvability condition at the next order. The first order problem is

$$\sqcup\psi_1 = F_1(x,z,t) \tag{4.7}$$

where F_1 is a functional of zeroth order terms. Solvability of (4.7) requires that

$$\int_0^{2\alpha_m} F_1\left[\frac{\varsigma - \chi}{2\alpha_m}, z, \frac{\varsigma + \chi}{2}\right] d\varsigma = 0 \tag{4.8}$$

This condition is derived is Bridges (1985). Application of (4.8) results in a partial differential equation for f,

$$\tau \frac{\partial^4 f}{\partial \chi^4} - \frac{6\omega_1}{\omega_0^3}\frac{\partial^2 f}{\partial \chi^2} - \frac{9}{\omega_0}\frac{\partial f}{\partial \chi}\frac{\partial^2 f}{\partial \chi^2} + 3\frac{\gamma}{\omega_0^4}\frac{\partial^2 f}{\partial z^2} = 0 \tag{4.9}$$

When $\gamma = 0$ the equation for f can be integrated to yield

$$f'(\chi) = A + B\mathrm{cn}^2(\chi; \kappa) \tag{4.10}$$

(where A and B are constants) the usual cnoidal wave and subsequent substitution into (4.5) results in a standing cnoidal wave (Bridges (1985)). With the retention of the $\frac{\partial^2 f}{\partial z^2}$ term the equation (4.10) is a form of the K-P equation. The K-P equation was first derived by Kadomtsev & Petviashvili (1970) in their study of the stability of solitary waves to transverse perturbations. A discussion and analysis of this equation can be found in Ablowitz & Segur (1979). Dubrovin (1981) and Segur & Finkel (1984) have shown that this equation is rich in the number of qualitatively different types of solutions which may be produced. Here the right and left running solutions of the K-P equation would be combined to form a weakly three dimensional standing wave. One example of a solution to (4.9) is an oblique travelling wave formed from $f(\chi, z) = f(\chi - \beta_n z)$.

Instead of analyzing this equation and its possibilities further an analysis with ξ unrestricted will be undertaken. With ξ unrestricted and $\delta = \sqrt{\tau\epsilon}$ a regular perturbation expansion in ϵ is assumed. Substitution into the governing equations and boundary conditions results in a wave equation in two space dimensions at leading order.

$$\omega_0^2 \frac{\partial^2 \psi_0}{\partial t^2} - \Delta\psi_0 = 0 \tag{4.11}$$

where

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \xi^2 \frac{\partial^2}{\partial z^2} \tag{4.12}$$

and it is required that the normal derivative of ψ_0 vanish at the vertical boundaries. The leading order wave height is given by

$$\eta_0(x, z, t) = -\omega_0 \frac{\partial}{\partial t}\psi_0(x, z, t) \tag{4.13}$$

The first order problem results in

$$\omega_0^2 \frac{\partial^2 \psi_1}{\partial t^2} - \Delta \psi_1 = \frac{\omega_0^4}{3} \frac{\partial}{\partial t} G_1(x, z, t) \tag{4.14}$$

where

$$G_1(x,z,t) = -\frac{3}{\omega_0^3}\left[\left(\frac{\partial \psi_0}{\partial x}\right)^2 + \xi^2\left(\frac{\partial \psi_0}{\partial z}\right)^2 + \frac{\omega_0^2}{2}\left(\frac{\partial \psi_0}{\partial t}\right)^2\right]$$

$$+ \tau \frac{\partial^3 \psi_0}{\partial t^3} - \frac{6\omega_1}{\omega_0^3}\frac{\partial \psi_0}{\partial t} \tag{4.15}$$

For solvability it is required that

$$\int_0^{2\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial \psi_0}{\partial t} G_1(x, z, t)\, dx\, dz\, dt = 0 \tag{4.16}$$

The function $\psi_0(x, z, t)$ which satisfies this functional differential equation is the leading order term for three dimensional standing waves in a rectangular basin when $\epsilon = O(\delta^2)$. A general solution to (4.16) has not been found. However it may be shown by substitution that with

$$\omega_1 = -2\tau\omega_0^3\left[\frac{\mathrm{E}(\pi;\kappa)}{\pi} - \frac{1}{3}(2 - \kappa^2)\right] \tag{4.17}$$

one possible expression for the leading order wave height which satisfies (4.16) is

$$\eta_0(x, z, t) = h(t + \alpha_m \bar{x} + \beta_n \bar{z}) + h(t - \alpha_m \bar{x} - \beta_n \bar{z}) + h(t - \alpha_m \bar{x} + \beta_n \bar{z}) + h(t + \alpha_m \bar{x} - \beta_n \bar{z}) \tag{4.18}$$

where

$$h(\rho) = A + B\mathrm{cn}^2(\rho;\kappa) \tag{4.19}$$

and

$$A = -\frac{4}{3}\omega_0^2\tau\{\kappa^2 - 1 + \frac{\mathrm{E}(\pi;\kappa)}{\pi}\} \tag{4.20}$$

$$B = \frac{4}{3}\kappa^2\tau\omega_0^2 \tag{4.21}$$

where $\mathrm{E}(\pi;\kappa)$, and $\mathrm{cn}(\rho;\kappa)$ are Jacobian elliptic functions (Byrd & Friedman (1971)). Periodicity in time and the finite domain require that κ^2, the modulus of the elliptic functions, satisfy the equation

$$\pi - \int_0^{\frac{\pi}{2}} \frac{d\varsigma}{\sqrt{1 - \kappa^2 \sin^2 \varsigma}} = 0 \tag{4.22}$$

Numerical evaluation results in $\kappa^2 \sim 0.9691$.

Therefore a solution of (4.16) is a set of four oblique travelling non-interacting (to leading order) cnoidal waves which when combined result in a nonlinear three dimensional standing cnoidal wave. It is also illuminating to note that the wave height (4.18) may be expressed as the infinite sum,

$$\eta_0(x,z,t) = k_0 \sum_{p=1}^{\infty} p a_p \cos p\alpha_m x \cos p\beta_n \bar{z} \cos pt \tag{4.23}$$

where k_0 is a constant and

$$a_0 = \frac{1}{\kappa^2}\left[\frac{\mathrm{E}(\pi;\kappa)}{\pi} + \kappa^2 - 1\right] \tag{4.24a}$$

$$a_p = \frac{2}{\kappa^2}\frac{pq^p}{1-q^{2p}} \quad for \quad p > 0 \tag{4.24b}$$

and $q = \exp[-\mathrm{K}(\frac{\pi}{2}; \sqrt{1-\kappa^2})]$.

Examples of the three dimensional cnoidal standing waves are given in Figures 3,4, and 5. Figure 3 is a $(m,n) = (1,1)$ mode at $t = 0$, Figure 4 is a $(m,n) = (1,2)$ mode at $t = 0$, and Figure 5 is a $(m,n) = (2,2)$ mode at $t = 0$. These figures give a prelude to the richness that is possible when a complete solution of (4.16) is found.

The solution obtained in (4.19) is only *one possible solution*. The analysis performed earlier for $\xi = 0(\epsilon)$ which resulted in a K-P type equation shows that *at least* in the region $\xi = 0(\epsilon)$ there are solutions other than the four travelling oblique cnoidal waves and it is expected that other regions of ξ will also have additional solutions.

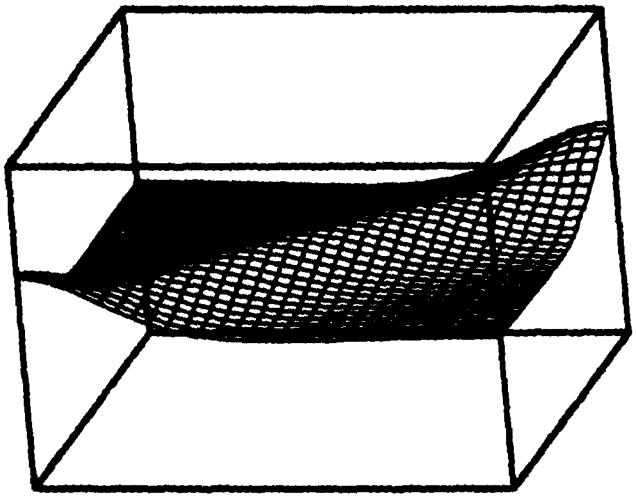

Figure 3. A 3-D cnoidal standing wave for $(m,n) = (1,1)$ and t=0.

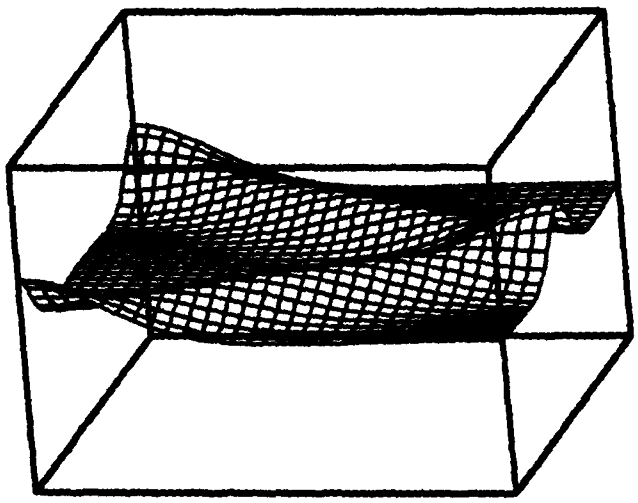

Figure 4. A 3-D cnodial standing wave for $(m,n) = (1,2)$ and t=0.

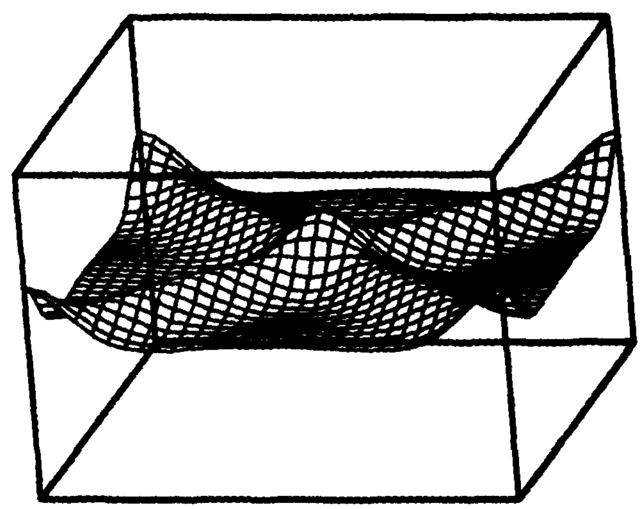

Figure 5. A 3-D cnoidal standing wave for $(m, n) = (2, 2)$ and $t=0$.

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This is an analysis of three dimensional surface waves which occur in a vessel of rectangular cross section and finite depth. The linearized problem has double eigenvalues for particular combinations of the parameters. It is shown that eight solution branches of finite amplitude are emitted by the double eigenvalues. Splitting of the double eigenvalues results in a secondary bifurcation. The direction of the emitted branches for the multiple and secondary bifurcation changes with the depth of the fluid. Finally it is shown that the formal solutions obtained are not uniformly valid and an

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ABSTRACT (continued)

additional expansion in the region of non-uniformity shows that the wave field changes type. One possibility in this region is a field of three-dimensional cnoidal standing waves.

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